

Black Hole Entropy: Microscopic vs. Microscopic*

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ABSTRACT: This is a brief review of the basic aspects of the black hole entropy in the viewpoint of general relativity and string theory.

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1. Review of Bekenstein-Hawking Entropy

1.1 Metric

Schwarzschild Black hole in 4D has the metric,

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 + r^2d\Omega^2, \quad (1.1)$$

with the Penrose digram (maximally continuation and compactification) in Figure 1, where I^+ is the future lightlike infinite. The black hole region can be formally defined as

$$B = M - J^-(I^+).$$

where $J^-(I^+)$ is the past of I^+ and M is whole spacetime manifold.

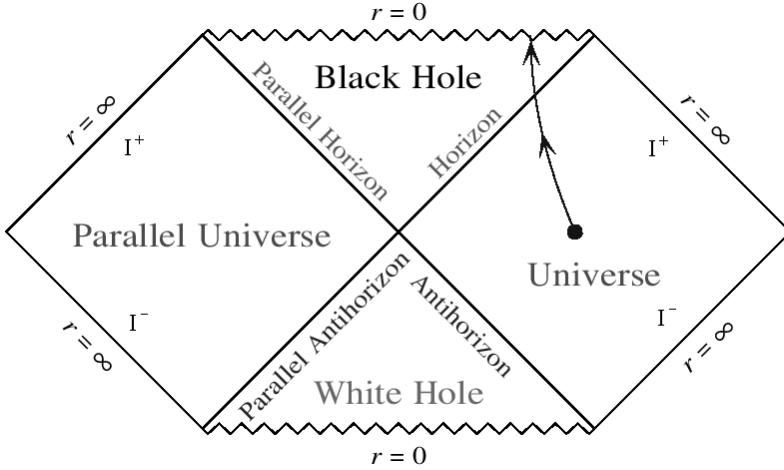


Figure 1: The Penrose diagram of Schwarzschild geometry

We can add the electric charge Q or the angular momentum J to get Kerr black hole or Reissner-Nordström black hole. If both Q and J are nonzero, the metric is Kerr-Newman black hole [2], ($G \equiv 1$),

$$ds^2 = -\frac{\Delta}{\rho^2}[dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2)d\phi - adt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \quad (1.2)$$

where $\Delta \equiv r^2 - 2Mr + a^2 + Q^2$, $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$ and $a = J/M$ which is the angular momentum per unit mass. We take the direction of \mathbf{J} along the z-axis. It generates the electromagnetic field,

$$\begin{aligned} F = & Q\rho^{-4}(r^2 - a^2 \cos^2 \theta)dr \wedge [dt - a \sin^2 \theta d\phi] \\ & + 2Q\rho^{-4}ar \cos \theta \sin \theta \wedge [(r^2 + a^2)d\phi - adt] \end{aligned} \quad (1.3)$$

When r is large, the electric field becomes the usual coulomb field of a charge Q and the magnetic field becomes the field of a magnetic dipole moment Qa .

Kerr-Newman black hole has the outer and inner horizon at,

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2 - a^2}. \quad (1.4)$$

As the case of Schwartzchild black hole, an observer can travel inside the horizon at $r = r_+$ but cannot send message out after crossing the outer horizon.

The cosmic censorship requires $M^2 \geq Q^2 + a^2$. When the equality holds, the corresponding black hole is called extremal. The two horizons coincides for an extremal black hole, $r_+ = r_- = r_0 = GM$,

$$ds^2 = -\left(\frac{r - r_0}{r}\right)^2 dt^2 + \left(\frac{r}{r - r_0}\right)^2 dr^2 + r^2 d\Omega_2^2. \quad (1.5)$$

or redefine $r \mapsto r - r_0$,

$$ds^2 = -\left(1 + \frac{r_0}{r}\right)^{-2} dt^2 + \left(1 + \frac{r_0}{r}\right)^2 (dr^2 + r^2 d\Omega_2^2), \quad (1.6)$$

so the near horizon limit is $AdS_2 \times S^2$. On the other hand, if $M^2 < Q^2 + a^2$, there is no horizon and the spacetime singularity is naked.

It is straightforward to generalize these solutions to higher dimension. For example, in 5D, the extremal black hole has the metric,

$$ds^2 = -\left[1 - \left(\frac{r_0}{r}\right)^2\right]^2 dt^2 + \left[1 - \left(\frac{r_0}{r}\right)^2\right]^2 dr^2 + r^2 d\Omega_3^2 \quad (1.7)$$

where $r_0^2 = 4G_5 M / 3\pi$.

1.2 Black hole thermodynamics

The black hole no hair theorem claims that in general relativity a 4D black hole is uniquely characterized by M , Q and J [3][4][5][6][7] and therefore the four types of black holes mentioned above are the only stationary black holes in 4D.¹ This surprisingly simple description is somewhat analogous to thermodynamics where a complex system with many degrees of freedom can be described by a small set of thermal variables macroscopically. So we may consider the black hole thermodynamics.

- The first law of black hole thermodynamics is *the conservation of energy*. The “thermodynamic variables” for a black hole would be the mass M , the angular momentum J , the charge Q and a variable to characterize the “size” which is chosen to be the horizon area A . For a Kerr-Newman black hole,

$$A = 4\pi(r_+^2 + a^2). \quad (1.8)$$

Take the derivative of Eqn.(1.8), we get [8]

$$dM = \Theta dA + \vec{\Omega} \cdot d\vec{L} + \Phi dQ, \quad (1.9)$$

where $\Theta = (r_+ - r_-)/4A$, $\vec{\Omega} = 4\pi\vec{a}/A$ and $\Phi = 4\pi Q r_+/A$. Comparing with the usual form of first law of thermodynamics,

$$dE = TdS - pdV$$

the last two terms of Eqn.(1.9) are the work done on the black hole: $\vec{\Omega}$ is the angular velocity and ϕ is the electric potential. Then we may tend to identify A as the entropy and Θ as the temperature.

¹For example, we may think that the magnetic moment of a black hole depends on the charge distribution. But by the no hair theorem, it is uniquely determined to be Qa .

- The second law claims that *the total entropy never decreases in a closed system*. So we need to (I) check that in a process which involves several black holes the total horizon area never decreases and (II) by the interaction of some known systems with a black hole to determine the entropy normalization.

Hawking proved that during the combination process of two Kerr black holes into one Kerr black hole, the horizon area never decreases [9].² Since the area A is never decreasing during the black hole combination process, A is proportional to the entropy S . Bekenstein considered massive particle capture process to roughly determine the normalization [8].³ The accurate ratio is determined by the accurate value of the black hole Hawking temperature [11],

$$T = \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2 + a^2}.$$

which gives

$$S = \frac{A}{4G} \quad (1.11)$$

which is the Bekenstein-Hawking temperature. The temperature can be obtained from the analytic continuation of the metric to the Euclidean spaces or by considering the entanglement entropy of a field in one universe and the parallel universe [12].

- The third law is that *as a system approaches absolute zero, the entropy approaches a minimum value*. Note that it does not mean that when $T \rightarrow 0$, the entropy must vanish like the perfect crystal. The extremal black hole has the zero temperature but nonzero entropy, which means there is a degeneracy about $\exp(S)$ of the ground states of a black hole. However, it is hard to realize this degeneracy in the classical physics.

In higher-dimensional space, the Bekenstein entropy formula is

$$S = \frac{A}{4G_D} \quad (1.12)$$

²The proof uses the fact that the black hole horizon is generated by the null geodesics without future end points [10]. Then we can analyze the null vector field on the horizon by Gauss theorem in curved space. A key assumption that the energy density is nonnegative everywhere, $T_{\mu\nu}n^\mu n^\nu \geq 0$ so that the divergence of the null vector field is positive or zero. Hence the final cross section of horizon with the final spacelike surface, which is just the final horizon area, must be larger or equal than the cross section of horizon with the initial spacelike surface, which is the initial horizon area.

³By absorbing a particle with the radius b and the mass μ the minimal horizon area increases by $\Delta A = 8\pi\mu b$. However, the particle's radius b cannot be zero by the uncertainty principle, $b \geq \mu/2\pi$, so the minimal increase of the horizon is, ($G = 1$)

$$\Delta A \geq 4. \quad (1.10)$$

The minimum increase of the entropy corresponds to the absorption of a bit information, which is $\log 2$, so the proportional factor is determined to be $S \sim \log 2A/4$.

where G_D is the D-dimensinonal Newton constant.

2. Rudiments of statistical mechanics

Statistical mechanics requires that the entropy in thermodynamics has a statistical explanation,

$$S = \log \Omega, \quad (2.1)$$

where in a microcanonical ensemble Ω is the number of microscopic states with the energy E .

The second law of thermodynamics is naturally the fact that a system tend to stay at a macroscopic state with relatively large number of microscopic states. In classical physics, like the ideal gas system, we know that although macroscopically the gas has only several thermodynamic variables, energy E , pressure p , volume V and the temperature T . But from the kinetic theory, a macroscopic system of gas contains huge number of molecules. Each molecule has the position x_i , momentum p_i and even the inner degrees of freedom. Even when a macroscopic variable, like E , is fixed, there are still huge number of combinations of each molecule's microscopic status. As the result, the entropy is large and may determine the system's behavior in classical physics.

However, for the black hole case, it is not easy to find the microscopic states by classical physics. An alternative explanation consider a free scalar field in the black hole geometry (fixed background) [13]. Then the entropy can be estimated by the quantum statistics and the Hawking temperature, like the usual photon entropy computation. However, this attempt gives divergent entropy which is in contradiction to the Bekenstein-Hawking entropy. A possible modification imposes a space cut-off with a Planck distance from the horizon and get the same order result of Bekenstein-Hawking entropy. This is a suggestion that near the horizon quantum gravity effect would be important and we need some new theory to treat the quantum states near horizon.

3. Black hole entropy in string theory

String theory give us new insights for black hole system. The massive objects, like D-branes warpped in extra dimensions, will generate black holes in noncompact dimension in the supergravity limit whose *macroscopic* entropy can be calculated classically by Bekenstin formula. On the other hand, the excitation of the objects, like the modes on a D-brane, are the microstates and whose degeneracy gives the *microscopic* entropy. It is interesting to see the consistency of the macroscopic and microscopic result.

Here is a subtlety: when we compactify the 10D string theory to get a black hole in lower dimensional spacetime, in general there are a lot of moduli fields. It would be a problem if the black hole entropy depends on the value of the moduli field at infinity, because where the spacetime is essentially flat and the value of moduli field are continuous but the microscopic entropy, $S = \log \Omega$, can only take discrete values because Ω is integer. This problem is avoid by *attractor mechanism* [15], which shows that the black hole entropy only depends on the moduli field value at the horizon and there is a attractor solution so the moduli value at the horizon is independent of its value at infinite.

Furthermore, since string theory has higher corrections beyond supergravity, there is a new type of black hole, “small black hole”, for which the horizon area in the supergravity limit is zero but its microscopic entropy is not zero. In this case, the configuration may have the entropy large in Planck length l_p but of the order one in string length l_s , so the supergravity is not no longer appropriate. String theory gives the generalizd Bekenstein entropy formula which is in consistency with the microscopic result. We would call the previous cases where the original Bekenstein formula is satisfied as “large black hole”.

4. Large black holes in string theory

Here we just consider the extremal black holes with zero temperature, because in this case we just need to count the degeneracy of the ground state. In particular, we focus on the *supersymmetric* black holes, because supersymmetry guranteed that the microscopic entropy obtained in weak coupling can be extrapolated to strong coupling region for supergravity.

The cases for $D > 5$ always give zero horizon area [19] because for a extremal black hole with finite horizon area in D dimension, the spacial metric behaves like

$$\left[1 + \left(\frac{r_0}{r}\right)^{D-3}\right]^{\frac{2}{D-3}} (dr^2 + r^2 d\Omega_{D-2}^2)$$

However, string theory constructions always give an integral for the exponential factor. If $D > 5$, then $2/(D-3)$ is not an integer so the extremal black hole cannot be obtained by string theory construction. Hence we just consider the cases $D = 4, 5$.

4.1 Five-dimensional black holes

For the five-dimensional black holes, at least three different charges are needed to get the nonzero horizon area. Strominger and Vafa consider a black hole obtained D1-D5 branes system wrapped on $K_3 \times S^1$ [14]. However, the original paper does not count the subtleties introduces by $K3$ in the macroscopic geometry, as noted by Johnson and Myers [16]. Alternative, we can consider a simpler case, a 5D black

hole from D1-D5 branes warped on $T^4 \times S^1$ with compact momentum [17][18]. We will review the latter case.

4.1.1 D1-D5 wrapped on $T^4 \times S^1$ with compact momentum

We consider the system with Q_1 D1-branes in the 5-direction and Q_5 D5-branes in the (5, 6, 7, 8, 9)-direction [20][21]. So they are in parallel with each other. Further, we assume that they coincide. To make the energy finite, we need the finite volume of branes, so we warp the (6, 7, 8, 9)-direction on a T^4 with the volume V_4 and the 5-direction on a S^1 with the length L . We also assume the system has a momentum p_5 on the compact 5-direction, so

$$p_5 = n_5/L,$$

where n_5 is an integral. So we have three different types of charges Q_1 , Q_5 and p_5 .

First, we consider the limit when $gQ \gg 1$, where Q would be any of the three charges. In this limit, the supergravity limit is a good approximation and the geometrical picture of a black hole is clear. By the analogy with black brane solution in supergravity, this system gives the black hole geometry (string frame),

$$ds^2 = Z_1^{-1/2} Z_5^{-1/2} [\eta_{\mu\nu} dx^\mu dx^\nu + (Z_n - 1)(dt + dx_5)^2] + Z_1^{1/2} Z_5^{1/2} dx^i dx^i + Z_1^{1/2} Z_5^{-1/2} dx^m dx^m \quad (4.1)$$

$$e^{-2\phi} = Z_5/Z_1. \quad (4.2)$$

We set the index μ, ν on the (0, 5)-direction, i, j on the (1, 2, 3, 4)-direction and m, n on the (6, 7, 8, 9) direction. The harmonic functions Z_1 , Z_5 and Z_n are defined to be

$$\begin{aligned} Z_1 &= 1 + \frac{r_1^2}{r^2}, \quad r_1^2 = \frac{(2\pi)^4 g Q_1 \alpha'^3}{V_4} \\ Z_5 &= 1 + \frac{r_5^2}{r^2}, \quad r_5^2 = g Q_5 \alpha' \\ Z_n &= 1 + \frac{r_n^2}{r^2}, \quad r_n^2 = \frac{(2\pi)^5 g^2 p_5 \alpha'^4}{L V_4} \end{aligned}$$

where $r^2 = x^i x^i$. Note that the horizon is at $r = 0$. In the viewpoint of the 5D noncompact spacetime (0, 1, 2, 3, 4), this solution is a black hole. Hence this coordinate just describes the geometry of the horizon and the spacetime outside the black hole.

The horizon area should be calculated in Einstein frame $ds_E^2 = e^{-\phi/2} ds^2$. The straightforward computation gives,

$$A = 8\pi G (Q_1 Q_5 n_5)^{1/2},$$

where G is the gravity constant in 5D. So the Bekenstein-Hawking entropy is

$$S = A/4G = 2\pi(Q_1 Q_5 n_5)^{1/2}, \quad (4.3)$$

Notice the result is dimensionless as it should for the entropy and has no dependence of the moduli like g , L and V_4 . It means that the change of such moduli is an adiabatic process.

The system is a BPS state: The original 10D string theory has 32 supersymmetry. The parallel D1-D5 system keeps 1/4 supersymmetry and the compact momentum breaks further 1/2 supersymmetry. So the system has 4 supersymmetry. The mass of this black hole is

$$M = p_5 + \frac{Q_1 L}{2\pi g_s \alpha'} + \frac{Q_5 L V}{g_s (2\pi)^5 \alpha'^3}, \quad (4.4)$$

which is simply the sum of Kaluza-Klein mass and the brane mass because it is a BPS state and there is no interaction energy.

Here is an interesting consequence of the discussion above. Suppose that we keep the number of each type of branes, but separate them: set the transverse coordinates of $Q_1^{(1)}$ D-1 branes at $(x^1, x^2, x^3, x^4) = (0, 0, 0, 0)$ while $Q_1^{(2)}$ D-1 branes at $(x^1, x^2, x^3, x^4) = (d, 0, 0, 0)$. We also set the $Q_5^{(1)}$ D-1 branes at $(x^1, x^2, x^3, x^4) = (0, 0, 0, 0)$ while $Q_5^{(2)}$ D-1 branes at $(x^1, x^2, x^3, x^4) = (d, 0, 0, 0)$. Fix $Q_1^{(1)} + Q_1^{(2)} = Q_1$ and $Q_5^{(1)} + Q_5^{(2)} = Q_5$, the total mass is not changed by Eq.(4.4). It seem that we separate a black hole into two black holes located at different positions. However, this process deceases the entropy by Eq.(4.3) so cannot happen.

Now we got the other limit $gQ \ll 1$ so the D-brane picture is available and we count the microscopic states of the D-branes. We assume that the change of picture does not change the number of microscopic states because of the supersymmetry. In D-brane picture, open strings which end on Q_1 D1 branes correspond to $U(Q_1)$ Yang-Mills fields and open strings which end on Q_5 D5 branes correspond to $U(Q_5)$ Yang-Mills fields. We need an action for the D1-D5 system to find the constraints of the field. The simplest way is to do T-dual on (1,2,3,4)-direction so we get Q_1 D5-branes and Q_5 D9-branes. This system has 8 supersymmetry so it is the $d = 6$, $N = 1$ super-Yang-Mills theory. The action is determined mainly by the supersymmetry. We just write down some related terms

$$L \supset |\partial_\alpha \chi + iA_\alpha \chi - i\chi A'_\alpha|^2 + \text{D-term}_{U(Q_1)} + \text{D-term}_{U(Q_5)},$$

where $\alpha = 0, 1, 2, 3, 4, 5$, A is $U(Q_1)$ gauge field and A' is $U(Q_5)$ gauge field. χ is a $Q_1 \times Q_5$ matrix whose element is a Weyl doublet which comes from the scalar field of hypermultiplet. For string theory viewpoint, χ is the massless open strings ends on both $D1$ and $D5$. The D-terms come from super-Yang-Mills theory and has the form $D^{Ai} D^{Ai}$

$$D^{Ai} = \chi^\dagger \sigma^A t^i \chi,$$

Then we do T-dual again on (1,2,3,4)-direction, so $A_i \rightarrow X_i/(2\pi\alpha')$ and $A'_i \rightarrow X'_i/(2\pi\alpha')$, we get the potential for $D1 - D4$ system,

$$V = \frac{1}{(2\pi\alpha')^2} |X_i\chi - \chi X'_i|^2 + \text{D-term}_{U(Q_1)} + \text{D-term}_{U(Q_5)}. \quad (4.5)$$

Now we have the potential for the moduli field of a 2D theory. Supersymmetry requires that $V = 0$. In general, there may be several different branches which correspond to different solutions. We need to determine which branch is the black hole state. If we take $\chi = 0$, then the first term in Eq.(4.5) vanishes so we can take arbitrary values of X_i and X'_i . However, X_i and X'_i are the transverse coordinates of the two types of D-branes so arbitrary value of X_i and X'_i will separate the branes and the original configuration is destroyed. So we consider the Higgs branch,

$$X_i = x_i I_{Q_1}, \quad X'_i = x'_i I_{Q_5}$$

In this case, again the the first term in Eq.(4.5) vanishes. We can count the number of free moduli spaces. X_m and A'_m provides $4Q_1^2 + 4Q_5^2$ degrees of freedom while χ provide $4Q_1 Q_5$. The D-term condition will remove $3Q_1^2 + 3Q_5^2$ degrees of freedom. The gauge symmetry will remove further $Q_1^2 + Q_5^2$. (They are rough estimation up to the quadratic term of Q .) So we have a 2D theory with $4Q_1 Q_5$ bosons. Supersymmetry provides further $4Q_1 Q_5$ fermions. The energy of this system is $E = n_5/L$.

In statistical mechanics, $S = \log n(E)$, where $n(E)$ is the degeneracy of states with fixed energy E . There are two ways to calculate $n(E)$:

- Consider the partition function $Z = \text{Tr}[\exp(-\beta H)]$. Suppose the 2D field theory is in finite temperature T , then the 0-direction. The partition function computation can be done by the standard 1-loop string worldsheet computation. (We just have left-handed modes because of the compact momentum.)

$$\text{Tr}[\exp(-\beta H)] = \sum_i q^{h_i - c/24}$$

where $q = \exp(-2\pi\beta/L)$ and c is the central charge. By modular invariance and take the limit $L \rightarrow \infty$, $Z = \exp(\pi c L / 12\beta)$, so

$$Z = \int_0^\infty dE n(E) \exp(-\beta H) = \exp(\pi c L / 12\beta).$$

This is a Laplacian transformation of $n(E)$. Reverse the transform, we get

$$n(E) \sim \exp[(\pi c L / 3)^{1/2}] = \exp[(2\pi)(Q_1 Q_5 n_5)^{1/2}].$$

where $E = n_5/L$ which means $S = \log n(E) = (2\pi)(Q_1 Q_5 n_5)^{1/2}$ which is consistent with Eq.(4.3).

- A more direct way is to compute the generating function of state degeneracy. If the system has the total compact momentum n_5 , the degeneracy is

$$\sum_{n_5} D(n_5) q^{n_5} = \left(\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} \right)^{4Q_1 Q_5}$$

where the $1+q^n$ is the Fermionic contribution and $1-q^n$ is the bosonic contribution. When n_5 is large, we get

$$D(n_5) \sim \exp(2\pi\sqrt{Q_1 Q_5 n_5})$$

since $E = n_5/L$ this gives the energy level degeneracy. It is also consistent with Eq.(4.3).

4.1.2 Formal analysis by 5D supergravity

We already see several examples of 5D large black holes in string theory and each of them contains three different types of the charges. The charges are coupled to the gauge fields in 5D supergravity, so it is possible to classify all the 5D large black holes by studying the 5D supergravity.

Recall that in type II string theories, before adding charges or compactifying the extra dimension on a Calabi-Yau manifolds, we have 32 supercharges which correspond to the $N=2$ supersymmetry in 10D. Compactify the 10D theory on T^5 , without breaking any supersymmetry, we get the $N=4$ supergravity in 5D. This theory contains 42 scalar field (moduli) which can be realized from the dimension reduction. The global symmetry is $G = E_{6,6}$ ⁴ while for a particular point in the moduli space the symmetry is $H = USp(8)$, so the moduli space is locally G/H .

5D supergravity with 32 supercharges also contains 27 $U(1)$ gauge fields, which are coupled with the charges in the previous examples. So there are 27 different types of charges, which can be embedded in the 8×8 antisymmetric central charge matrix A , with the constraint,

$$Tr(\Omega A) = 0,$$

where Ω is the antisymmetric symplectic metric with $\Omega_{12} = \Omega_{34} = \Omega_{56} = \Omega_{78} = 1$. The global symmetry G is acting on Z and the action by the subgroup H is manifest,

$$Z \mapsto A^T Z A \tag{4.6}$$

where $A \in USP(8)$, so $A^T \Omega A = \Omega$.

The black hole entropy should be a function of the 27 charges, or in the other word, a function of Z . (Here we already assumed that the entropy is independent of the moduli, like the radius of torus. See the section on attractor mechanism.) The

⁴This global symmetry is broken for the whole string theory and only $E_6(\mathbb{Z})$ is preserved.

entropy should also be invariant under the global symmetry G , in particular, (4.6), and a cubic function of the charges, as we show before. The unique entropy formula up to the normalization is,

$$S = 2\pi\sqrt{\Delta}, \quad \Delta = -\frac{1}{48}Tr(\Omega Z \Omega Z \Omega Z). \quad (4.7)$$

This formal expression can be reduced to the explicit form like (4.3). (4.6) would transfer the matrix Z into the form such that $Z_{12} = x_1$, $Z_{34} = x_2$, $Z_{56} = x_3$ and $Z_{78} = x_4$ while $x_1 + x_2 + x_3 + x_4 = 0$ because of the constraint. Resolve the constraint by,

$$\begin{aligned} x_1 &= Q_1 - Q_2 - Q_3 \\ x_2 &= -Q_1 + Q_2 - Q_3 \\ x_3 &= -Q_1 - Q_2 + Q_3 \\ x_4 &= Q_1 + Q_2 + Q_3 \end{aligned}$$

We have $S = 2\pi\sqrt{Q_1 Q_2 Q_3}$ which has the same form as (4.3).

4.2 Four-dimensional black holes

For the four-dimensional black holes, the situation is more complicated since we need at least four different charges to get nonzero horizon area.

4.2.1 Formal analysis by 4D supergravity

The similar formal analysis based on supergravity still works. In this case, the 10D IIB supergravity is reduced to $D = 4$, $N = 8$ supergravity, which contains 70 scalar fields and 28 vector fields. The global symmetry is $G = E_{7,7}$ while $H = SU(8)$. The corresponding 28 charges are again embedded in the 8×8 central charge matrix, but without constraint. The G -invariant quartic function of the charges is,

$$\Delta = Tr(Z \bar{Z} Z \bar{Z}) - \frac{1}{4}(Tr(Z \bar{Z}))^2 + 4(PfZ + Pf\bar{Z})$$

where PfZ is the Pfaffian of Z . Again, we rotate Z by a $SU(8)$ matrix U , $Z \mapsto U^T Z U$, and resolve the resulting element by the charge Q 's, we get

$$\Delta = Q_1 Q_2 Q_3 Q_4,$$

and the entropy has the form $S = 2\pi \log \sqrt{\Delta}$.

4.2.2 D2-D6-NS5 brane with compact momentum

4.2.3 D3 brane warpped on three-cycle

Type IIB string theory compactified on Calabi-Yau manifold gives $D = 4$, $N = 2$ supersymmetry containing $h^{1,1} + 1$ vector multiplets which are the complex structure

moduli. A 4-dimensional black hole can be realized by wrapping D3-branes on a *special Lagrangian three-cycle*⁵ C . [22]

The wrapped D3-brane has to satisfy the BPS bound,

$$M \geq e^{K/2} \left| \int_C \Omega \right|. \quad (4.8)$$

where $K = -\log(i \int_M \Omega \wedge \bar{\Omega})$ is the Kähler potential. We can use the special geometry to rewrite the bound,

$$X^I = e^{K/2} \int_{A^I} \Omega, \quad F_I = e^{K/2} \int_{B_I} \Omega, \quad I = 0, \dots, h^{1,1} \quad (4.9)$$

and the normalization of Ω can be avoided by defining

$$t^\alpha = \frac{X^I}{X^0}, \quad \alpha = 1, \dots, h^{1,1}$$

On the basis of 3-cycles, the supersymmetric cycle is expanded as $C = p^I B_I - q_I A^I$ and its dual is $\Gamma = p^I \alpha_I - q_I \beta^I$. So the BPS bound can be rewritten as,

$$M \geq |p^I F_I - q_I X^I|.$$

We just consider the extremal case, $M = |p^I F_I - q_I X^I| \equiv |Z|$, then it seems that the black hole metric is (1.6) with $M = |Z|$. However, this argument is naive because we did not fix the moduli fields. Without breaking the spherical symmetry, the metric in general is,

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} (dr^2 + r^2 d\Omega_2^2) \quad (4.10)$$

where the $U(r)$ is related to the r -dependent moduli fields, $t^\alpha(r)$. Note that when $r \rightarrow \infty$, the spacetime is asymptotically flat so the moduli at infinity, $t^\alpha(r = \infty)$, are free. If the horizon area of (4.10) depends on the continuous values of $t^\alpha(r = \infty)$, it would be a contradiction since by Bekenstein formula,

$$A = 4S = \log \Omega$$

the horizon area can only takes discrete values. This problem is solved by the attractor mechanism [15].

⁵A special Lagrangian manifold is a submanifold such that some supersymmetry is preserved when on it a D-brane or M-brane is wrapped. By the explicit form of supersymmetric transformation, the conditions for such submanifold are

$$i^* J = 0, \quad i^* \Omega \propto v$$

where i^* is the pullback from the Calabi-Yau manifold to the submanifold, J and Ω are the Kähler form and the (3,0) form of the Calabi-Yau manifold and v is the volume form of the submanifold.

5. Attractor Mechanism

We consider the metric (4.10) and the r -dependent moduli $t^\alpha(r)$ by the method [23]. Instead of solving the Einstein equation for $U(r)$, which is a second order differential equation, we use the BPS equation (first order equation) for unbroken supersymmetry,

$$\frac{dU}{d\tau} = -e^U |Z|, \quad (5.1)$$

$$\frac{dt^\alpha}{d\tau} = -2e^U G^{\alpha\bar{\beta}} \partial_{\bar{\beta}} |Z| \quad (5.2)$$

which correspond to $\delta\psi_\mu = 0$ and $\delta\lambda^\alpha = 0$. Here we inverted the radius coordinate by defining $\tau = 1/r$, so the horizon is at $\tau = \infty$ while the asymptotically flat region is at $\tau = 0$. $|Z|$ is the absolute value of the central charge,

$$Z(\Gamma) \equiv e^{i\alpha} |Z| = e^{K/2} \int_C \Omega.$$

The equations 5.2 gives a flow on the moduli space, for which the initial point is the moduli's value in the asymptotically flat region and the ending point is at the near horizon region. To solve the moduli problem, we want to show that,

- There is an attract point in the moduli space so that the different flows will end in the same point. In the other word, the value $t^\alpha(\tau = \infty)$ is fixed, independent of the initial value, $t^\alpha(\tau = 0)$.
- The horizon area only depends on $t^\alpha(\tau = \infty)$ and $U^\alpha(\tau = \infty)$. So the black hole entropy is independent of the moduli in the asymptotically flat region.

First, we show that the equations (5.2) has the analogous form of the damped oscillator, so the attractor exist. $|Z|$ is a function of τ , via the moduli field, $|Z|(\tau) = |Z|(t^\alpha(\tau), \bar{t}^\beta(\tau))$, so

$$\frac{d|Z|}{d\tau} = \frac{dt^\alpha}{d\tau} \partial_\alpha |Z| + \frac{d\bar{t}^\alpha}{d\tau} \partial_{\bar{\alpha}} |Z| = -4e^U G^{\alpha\bar{\beta}} \partial_\alpha |Z| \partial_{\bar{\beta}} |Z| \leq 0, \quad (5.3)$$

So $|Z|$ is monotonically decreasing during the evolution, just like the energy for the damped oscillation, which implies the existence of an attractor. See the table (1) for the detail.

In general, when $\tau \rightarrow \infty$, the flow ends up at a point S in the moduli space where $|Z|$ takes the finite minimal value $|Z_*$, since $|Z|$ is bounded below. The point S is an attractor since $|Z|$ is monotonically decreasing during the evolution. No matter where the initial point is in the moduli space, the final (near horizon) value

	Damped oscillator	Moduli fields
Coordinates	p, x	t^α
Time	t	τ
E.O.M	$\frac{dx}{dt} = \frac{p}{m}, \frac{dp}{dt} = -kx - \frac{\gamma p}{m}$	$\frac{dt^\alpha}{d\tau} = -2e^U G^{\alpha\bar{\beta}} \partial_{\bar{\beta}} Z $
Energy	$E = \frac{p^2}{2m} + \frac{kx^2}{2} \geq 0$	$ Z \geq 0$
Energy evolution	$\frac{dE}{dt} = -\frac{\gamma p^2}{2m^2} \leq 0$	$\frac{d Z }{d\tau} = -4e^U G^{\alpha\bar{\beta}} \partial_\alpha Z \partial_{\bar{\beta}} Z \leq 0$
Attractor	minimal E	minimal $ Z $

Table 1: the attractor mechanism as an analogy as the damped oscillator

of the moduli fields is fixed as S . Furthermore, for large τ , the moduli fields are approaching the fixed point, $|Z| \sim |Z_*$, so (5.1) reads,

$$e^{-U} \sim \tau |Z_*|$$

and the near-horizon metric is $AdS_2 \times S^2$,

$$ds^2 \sim -\frac{r^2}{|Z_*|^2} dt^2 + \frac{|Z_*|^2}{r^2} (dr^2 + r^2 d\Omega_2^2),$$

with the horizon area,

$$A = 4\pi |Z_*|^2 \quad (5.4)$$

which has no dependence of the moduli field at infinity.

(5.1) and (5.2) can be combined into one equation,

$$2 \frac{d}{d\tau} \left[e^{-U+K/2} Im(e^{-i\alpha} \Omega) \right] = -\Gamma + \text{exact form} \quad (5.5)$$

In the near horizon limit,

$$2e^{K/2} Im(\bar{Z}_* \Omega) = -\Gamma + \text{exact form}$$

which means the dual form of the supersymmetric cycle is combination of a $(3,0)$ form and a $(0,3)$ form near horizon. This characterized the attractor point. The integral over A^I and B_J gives the attractor equation in special geometry formula,

$$p^I = -2Im(\bar{Z} X^I) \quad \text{and} \quad q_I = -2Im(\bar{Z} F_I) \quad (5.6)$$

6. Small black holes in string theory* not complete!

The analysis on black hole entropy are based on general relativity or supergravity. However, we think that there would be a quantum gravity theory for which supergravity is an approximation. We know that in string theory, string interaction [21]

would produce higher order curvature terms in additional to the original Einstein-Hilbert action. Does the higher curvature terms change the Bekenstein-Hawking black-hole formula?

Wald [24] gives a more general black hole entropy formula for a general gravity Lagrangian which can be a function of $g_{\mu\nu}$, $R_{\mu\nu\rho\lambda}$, $D_\tau R_{\mu\nu\rho\lambda}$, etc,

$$S = 2\pi \int_{horizon} d\Omega \epsilon_{\mu\nu} \epsilon_{\mu\nu} \frac{\partial L}{\partial R_{\mu\nu\rho\lambda}} \quad (6.1)$$

where μ and ν are the index of 2D horizon coordinates. This formula is derived by the Noether charge on horizon. When the gravity theory is general relativity, $L = R/16G$, the Eq.(6.1) is reduced to $S = A/4G$ which is Bekenstein-Hawking entropy. However, with the higher curvature corrections, even when the horizon has a zero area, the entropy can still be nonzero by Wald's entropy formula.

Here we consider an example in string theory [25]. The heterotic string is compactified on $T^4 \times T^2$ where T^4 is in (6,7,8,9)-direction and T^2 is in (4,5)-direction. Consider a string with winding number w along the 5-direction. For the heterotic string compactified on T^6 , we have the $O(22, 6; \mathbb{Z})$ symmetry while the internal momenta p_L has 22 components and p_R has 6 components. The even lattice condition reads (in the dimensionless unit)

$$p_R^2 - p_L^2 \text{ is even}$$

and the Virasoro constraint gives,

$$\frac{1}{4}\alpha' m^2 = \frac{1}{2}p_R^2 + N_R = \frac{1}{2}p_L^2 + N_L - 1.$$

For the (n, w) state $p_R^2 - p_L^2 = 2nw$. We want to make the (n, w) state to be a BPS state, so we set $N_R = 0$ to keep half of the supersymmetry.

$$N_L = 1 - nw. \quad (6.2)$$

The number of states is generated by the partition function,

$$Z(\beta) = 16 \sum d_N \exp(-\beta N) = \frac{16}{\Delta(q)^{24}}$$

where $N = w|n| = N_L - 1$. The factor 16 comes from the degeneracy of the right-moving supersymmetric ground state and latter equality comes from the 24 left-handed bosons. $\Delta(q) = \eta(q)^{24}$

Again, we use the inverse Laplacian transformation and by the modular invariance,

$$d_N = \frac{1}{2\pi i} \int_C d\beta \exp(\beta N) \left(\frac{\beta}{2\pi}\right)^{12} \frac{1}{\Delta(e^{-4\pi^2/\beta})}. \quad (6.3)$$

The saddle point approximation gives $d_N \sim \exp(4\pi\sqrt{w|n|})$. More accurate result for large N is given in [19]

$$d_N \sim 16\hat{I}_{13}(4\pi\sqrt{N}).$$

where the special function is

$$\hat{I}_\nu(z) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \left(\frac{t}{2\pi}\right)^{-\nu-1} e^{t+z^2/4t} dt$$

As before, the statistical entropy is $S = \log d_N$. So we may like to find the space-time geometry of this configuration by supergravity as before. Then we may expect that the Bekenstein-Hawking entropy from the horizon area will be consistent with the statistical entropy computed above. However, when N is large, the supergravity approximation will break down and the high curvature terms will be important. The supergravity solution is obtained but gives zero horizon area. [26] (Actually, this model just have 2 charges in 4D but nonzero 4D black hole horizon need 4 charges.) So we need to find the geometry and takes the high curvature terms into account.

It is more convient to use the dual picture, type IIA compactified on $K_3 \times T^2$. The special geometry is applicable and the quantum gravity correction can be read from the prepotential. The system has $D = 4$ $N = 4$ supersymmetry, so only the first correction to the prepotential is nonvanishing. Here we still use the $N = 2$ complex special geometry. The vector multiplet moduli space of $N = 2$ supergravity with n_v vector multiplets can be parameterized by $n_v + 1$ complex projective coordinates X^I . There are infinite numbers of higher derivative corrections to the Einstein-Hilbert action. The holomorphic prepotential $F(X^I)$ is generalized to be

$$F(X^I, W^2) = \sum_{h=0}^{\infty} F_h(X^I) W^{2h}$$

where h is the genus and W is the reduced chiral multiple that involves the graviphoton field strength. The prepotential satisfies the homogeneity relation,

$$X^I \partial_I F(X^I, W^2) + W \partial_W F(X^I, W^2) = 2F(X^I, W^2)$$

The attractor point in the moduli spaces is determined by the equations

$$p^I = \text{Re}[CX^I] \tag{6.4}$$

$$q_I = \text{Re}[CF_I] \tag{6.5}$$

where p^I is the magnetic charge and q_I is the electric charge. By the convention, the graviphoton field strength at the horizon takes the value $C^2 W^2 = 256$. $F_I \equiv \partial F / \partial X^I$. The high order correction for the supergravity is given in [27]. By the Eq.(6.1),

$$S = \frac{\pi i}{2} (q_I \bar{C} \bar{X}^I - p^I \bar{C} \bar{F}_I) + \frac{\pi}{2} \text{Im}[C^3 \partial_C F].$$

The first equation Eq.(6.4) can be solved as,

$$CX^I = p^I + \frac{i}{\pi} \phi^I$$

We define the “free energy” to be

$$\mathcal{F} = -\pi \text{Im} \left[F(p^I + \frac{i}{\pi} \phi^I, 256) \right]$$

so

$$q_I = \frac{1}{2}(CF_I + \bar{C}\bar{F}_I) = -\frac{\partial}{\partial \phi^I} \mathcal{F}(\phi, p).$$

The Wald entropy can be rewrite as

$$S(q, p) = \mathcal{F}(\phi, p) - \phi^I \frac{\partial}{\partial \phi^I} \mathcal{F}(\phi, p).$$

So the entropy of a black hole is the Legendre transformation of \mathcal{F} with respect to ϕ^I . ϕ^I play the role of chemical potentials. By statistical mechanics, the “partition function”

$$\mathcal{Z}(\phi^I, p^I) = e^{\mathcal{F}(\phi^I, p^I)} = \sum_{q^I} \Omega(q_I, p^I) e^{-\phi^I q_I}.$$

Now we return to the (n, w) system. It is dual to the type IIA theory on $K3 \times T^2$ with w D4-brane wrapped on the K3 and n D0-branes. The D0-branes are electrically charged and the D4-brane is magnetically charged. So only the two charges, $q_0 = n$ and $p^1 = w$ are nonzero. Since the theory has $N = 4$ supersymmetry, the only nonvanishing contributions to the prepotential are F_0 and F_1 . The free energy is [26],

$$\mathcal{F} = -\frac{1}{2} C_{ab} \phi^a \phi^b \frac{p^1}{\phi^0} - \log(|\Delta(q)|^2)$$

with

$$q = \exp\left(\frac{2\pi^2 p^1}{\phi^0} + \frac{2\pi i \phi^1}{\phi^0}\right).$$

Then the attractor equations can be solved as $q^A = (q^0, 0, \dots, 0)$ and $p_A = (0, p_1, 0, \dots, 0)$. By the Legendre transformation,

$$\phi^0 = -2\pi \sqrt{\frac{p^1}{|q_0|}}$$

and the entropy

$$S = 4\pi \sqrt{p^1 |q_0|} = 4\pi \sqrt{w |n|}.$$

This “thermodynamic” entropy is consistent with the statistical result Eq.(6.3) in the saddle point approximation.

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